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PERVERSE NORI MOTIVES

by

Florian Ivorra

Abstract. — Let $k = \mathbb{C}$ be the field of complex numbers (one can also choose a field of characteristic zero k with a fixed embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$). Assume that K is a field. In this work, we show that the Tannakian formalism developed by M. NORI also applies to representations $T : \mathcal{Q} \rightarrow \mathcal{P}$ with values in a K -linear Abelian category \mathcal{P} which is Noetherian, Artinian and has finite dimensional Hom groups over K . As an application, we define a relative version, modeled after perverse sheaves, of the Abelian category of motives constructed by M. NORI over k .

1. Introduction

Let $k = \mathbb{C}$ be the field of complex numbers (one can also choose a field of characteristic zero k with a fixed embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$). In this work, a k -variety will be a quasi-projective k -scheme.

1.1. Using a Tannakian approach, MADHAV NORI has defined a category of mixed motives $\text{NMM}(k)$ over k . Though M. NORI himself did not publish his construction, several accounts of it are available in the literature (see e.g. [13, 23, 10]). Roughly speaking, $\text{NMM}(k)$ is the universal Abelian category having a *faithful exact* Betti realization in the category of finite dimensional \mathbb{Q} -vector spaces and a relative homology theory for pairs consisting of a k -variety and a closed subscheme of it. Being a motive in $\text{NMM}(k)$ is the best structure that one can put on the relative homology of a pair of k -varieties. In particular, as the relative homology of pairs carries a (polarizable) mixed Hodge structure, the Betti realization factors through the Abelian category $\text{MHS}_{\mathbb{Q}}^p$ of polarizable mixed \mathbb{Q} -Hodge structures i.e. one has a faithful exact functor

$$\text{NMM}(k) \rightarrow \text{MHS}_{\mathbb{Q}}^p.$$

1.2. Assume that K is a field. In this work, we show that the Tannakian formalism developed by M. NORI also applies to representations $\mathbb{T} : \mathcal{Q} \rightarrow \mathcal{P}$ with values in a K -linear Abelian category \mathcal{P} which is Noetherian, Artinian and has finite dimensional Hom groups over K . The proof of this result is an application of Nori's statement and the characterization of categories of comodules over K -coalgebras obtained by M. TAKEUCHI in [20].

1.3. As an application, we develop a relative version, modeled after perverse sheaves, of the Abelian category of motives constructed by M. NORI. More precisely, for every quasi-projective k -scheme X , we construct a \mathbb{Q} -linear Abelian category $\mathcal{N}(X)$ of “perverse motives”, with a faithful exact functor

$$\mathrm{rat}_X^{\mathcal{N}} : \mathcal{N}(X) \rightarrow \mathcal{P}(X)$$

to the category $\mathcal{P}(X)$ of perverse sheaves over X . The category $\mathcal{N}(\mathrm{Spec}(k))$ is the category $\mathrm{NMM}(k, \mathbb{Q})$ of M. NORI obtained by considering homology with rational coefficients. The Betti functor $\mathrm{rat}_X^{\mathcal{N}}$ factors through the Abelian category $\mathrm{MHM}(X, \mathbb{Q})$ of mixed Hodge modules constructed by M. SAITO in [17, 18] and this provides a faithful exact functor

$$\mathcal{N}(X) \rightarrow \mathrm{MHM}(X, \mathbb{Q}).$$

1.4. The category $\mathrm{NMM}(k)$ contains more motives, than expected a priori from its definition: (a) M. NORI constructs a realization functor (see e.g. [13] for a sketch of the construction)

$$\mathrm{DM}_{gm}(k) \rightarrow \mathrm{D}^b(\mathrm{NMM}(k))$$

where $\mathrm{DM}_{gm}(k)$ is the triangulated category of geometric motives constructed by V. Voevodsky in [22]; (b) in [1, Theorem 3.1], D. ARAPURA shows that the Leray spectral sequence for Betti cohomology associated with a *projective* morphism $f : X \rightarrow Y$ of quasi-projective k -varieties is motivic in the sense of Nori i.e. is the image of a spectral sequence in the category $\mathrm{NMM}(k)$ via the Betti realization functor.

In [11] we show that, for every smooth quasi-projective k -scheme X , the derived category $\mathrm{D}^b(\mathcal{N}(X))$ is the target of a realization functor

$$\mathbf{DA}_{\mathrm{ct}}^{\mathrm{\acute{e}t}}(X, \mathbb{Q}) \rightarrow \mathrm{D}^b(\mathcal{N}(X))$$

where $\mathbf{DA}_{\mathrm{ct}}^{\mathrm{\acute{e}t}}(X, \mathbb{Q})$ is the triangulated category of étale constructible motives with rational coefficients introduced by J. AYOUB in [3, 4]. The category $\mathbf{DA}^{\mathrm{\acute{e}t}}(X, \mathbb{Q})$ is the \mathbb{Q} -linear étale counterpart of the stable homotopy category of X -schemes of F. MOREL and V. VOEVODSKY (see [12, 15, 21]). By [5, Théorème B.1] and [14, Theorem 14.30, Lemma 14.21], it is known that the categories $\mathrm{DM}_{gm}(k, \mathbb{Q})$ and $\mathbf{DA}_{\mathrm{ct}}^{\mathrm{\acute{e}t}}(k, \mathbb{Q})$ are equivalent.

1.5. Note that a higher dimensional analog of Nori's construction, based on constructible sheaves rather than perverse sheaves, has been developed by D. ARAPURA in [2]. The category $\mathcal{M}(X, \mathbb{Q})$ constructed by D. ARAPURA in *loc.cit.* is probably related to the category $\mathcal{N}(X)$ that we construct in the present work. By analogy with perverse sheaves and mixed Hodge modules, it may be possible that $\mathcal{M}(X, \mathbb{Q})$ is

equivalent to the heart of a certain t -structure on $D^b(\mathcal{N}(X))$ but we do not attempt in this work to prove such a comparison result.

2. Statement of the result

2.1. Let \mathfrak{D} be a strict 2-category. We will denote the composition of 1-morphisms (resp. 2-morphisms) by the symbol \circ . Given a diagram of the form

$$\begin{array}{ccccc} & f & & k & \\ X & \xrightarrow{\quad} & Y & \xrightarrow{\quad} & Z \\ & \Downarrow \alpha & & \Downarrow \beta & \\ & g & & h & \end{array}$$

we will denote by $\beta \star \alpha : h \circ f \rightarrow k \circ g$ the so-called horizontal composition of the 2-morphisms α and β . If $\alpha = \text{id}$, we will simply write $f \star \beta = \text{id} \star \beta$ and similarly if $\beta = \text{id}$ we will write $k \star \alpha = \alpha \star \text{id}$. Given two objects $X, Y \in \mathfrak{D}$, we denote by $\mathfrak{D}(X, Y)$ the category formed by the 1-morphisms from X to Y . Let us fix some terminology:

- Given objects $X, Y \in \mathfrak{D}$, we say that the property **I**(X, Y) holds if the category $\mathfrak{D}(X, Y)$ is non empty and every object of $\mathfrak{D}(X, Y)$ is initial.
- Given an object $X \in \mathfrak{D}$, we say that X has the property **I** if the property **I**(X, Y) holds for any object Y in \mathfrak{D} .

An object X that has the property **I** will be called 2-initial. Such an object in \mathfrak{D} is unique up to an equivalence which is unique up to a unique invertible 2-morphism. Indeed if X and X' are two objects in \mathfrak{D} that are 2-initial. Then there exists a 1-morphism $f : X \rightarrow X'$ and any such 1-morphism is an equivalence. Moreover given two 1-morphisms $f, g : X \rightarrow X'$, there exists one and only 2-morphism $\alpha : f \Rightarrow g$ and this 2-morphism is invertible.

Remark 2.1. — If Y and Y' are equivalent in \mathfrak{D} , then the category $\mathfrak{D}(X, Y)$ and $\mathfrak{D}(X, Y')$ are equivalent. In particular the property **I**(X, Y) holds if and only if the property **I**(X, Y') holds.

2.2. Let \mathscr{P} be a K -linear Abelian category and $T : \mathcal{Q} \rightarrow \mathscr{P}$ be a representation. With this representation is associated a strict 2-category \mathfrak{D}_T defined as follows.

- An object in \mathfrak{D}_T is a 4-uplet $(\mathcal{A}, R, F, \alpha)$ where \mathcal{A} is a K -linear abelian category, $R : \mathcal{Q} \rightarrow \mathcal{A}$ is a representation, $F : \mathcal{A} \rightarrow \mathscr{P}$ is a K -linear faithful exact functor and $\alpha : F \circ R \rightarrow T$ is an invertible 2-morphism.
- If $(\mathcal{A}, R, F, \alpha)$ and $(\mathcal{B}, S, G, \beta)$ are two objects, a 1-morphism $(\mathcal{A}, R, F, \alpha) \rightarrow (\mathcal{B}, S, G, \beta)$ is a triplet (H, γ, δ) where $H : \mathcal{A} \rightarrow \mathcal{B}$ is a K -linear functor and

$$\gamma : H \circ R \xrightarrow{\sim} S; \quad \delta : G \circ H \xrightarrow{\sim} F$$

are invertible 2-morphisms such that the square

$$\begin{array}{ccc} G \circ H \circ R & \xrightarrow{G \star \gamma} & G \circ S \\ \downarrow \delta \star R & & \downarrow \beta \\ F \circ R & \xrightarrow{\alpha} & T \end{array}$$

is commutative.

- If (H, γ, δ) and (H', γ', δ') are 1-morphisms between the same two objects, a 2-morphism $(H, \gamma, \delta) \Rightarrow (H', \gamma', \delta')$ is a 2-morphism⁽¹⁾ $\theta : H \rightarrow H'$ such that $\gamma' \circ (\theta \star R) = \gamma$ and $\delta' \circ (G \star \theta) = \delta$.

Definition 2.2. — We say that the category \mathcal{P} has the property **N** if, for every quiver \mathcal{Q} and every representation $T : \mathcal{Q} \rightarrow \mathcal{P}$, the 2-category \mathfrak{D}_T has a 2-initial object.

This means that there exist a K -linear Abelian category \mathcal{A} , a representation $R : \mathcal{Q} \rightarrow \mathcal{A}$, a K -linear faithful exact functor $F : \mathcal{A} \rightarrow \mathcal{P}$ and an invertible 2-morphism $\alpha : F \circ R \rightarrow T$ such that for every K -linear Abelian category \mathcal{B} , every representation $S : \mathcal{Q} \rightarrow \mathcal{B}$, every K -linear exact faithful functor $G : \mathcal{B} \rightarrow \mathcal{P}$, and every invertible 2-morphism $\beta : G \circ S \rightarrow T$ the following conditions are satisfied.

- There exists a K -linear functor $H : \mathcal{A} \rightarrow \mathcal{B}$ and two invertible 2-morphisms

$$\gamma : H \circ R \xrightarrow{\sim} S; \quad \delta : G \circ H \xrightarrow{\sim} F$$

such that the square

$$\begin{array}{ccc} G \circ H \circ R & \xrightarrow{G \star \gamma} & G \circ S \\ \downarrow \delta \star R & & \downarrow \beta \\ F \circ R & \xrightarrow{\alpha} & T \end{array}$$

is commutative.

- If $H' : \mathcal{A} \rightarrow \mathcal{B}$ is a K -linear functor and

$$\gamma' : H' \circ R \xrightarrow{\sim} S; \quad \delta' : G \circ H' \xrightarrow{\sim} F$$

are two invertible 2-morphisms such that the square

$$\begin{array}{ccc} G \circ H' \circ R & \xrightarrow{G \star \gamma'} & G \circ S \\ \downarrow \delta' \star R & & \downarrow \beta \\ F \circ R & \xrightarrow{\alpha} & T \end{array}$$

is commutative, then there exists a unique 2-morphism $\theta : H \rightarrow H'$ such that $\gamma' \circ (\theta \star R) = \gamma$ and $\delta' \circ (G \star \theta) = \delta$.

Let us fix a quiver \mathcal{Q} and a representation $T : \mathcal{Q} \rightarrow \mathcal{P}$. Note that a 2-morphism θ in \mathfrak{D}_T is invertible in \mathfrak{D}_T if and only if it is invertible as a natural transformation. A similar result holds for 1-morphism. More precisely:

Lemma 2.3. — Let $(\mathcal{A}, R, F, \alpha)$ and $(\mathcal{B}, S, G, \beta)$ be two objects in \mathfrak{D}_T . A 1-morphism

$$(H, \gamma, \delta) : (\mathcal{A}, R, F, \alpha) \rightarrow (\mathcal{B}, S, G, \beta)$$

is an equivalence in \mathfrak{D}_T if and only if H is an equivalence of categories.

⁽¹⁾i.e. a natural transformation.

Proof. — If (H, γ, δ) is an equivalence in \mathfrak{D}_T , then it follows from the definition that H is an equivalence of categories. Assume now that H is an equivalence. There exist a K -linear functor $H' : \mathcal{B} \rightarrow \mathcal{A}$, an invertible 2-morphism $\phi : H \circ H' \rightarrow \text{id}_{\mathcal{B}}$ and an invertible 2-morphism $\psi : H' \circ H \rightarrow \text{id}_{\mathcal{A}}$ such that $H' \star \phi = \psi \star H'$ and $H \star \psi = \phi \star H$. Consider the 2-morphisms ε, η obtained respectively by the composition of the following sequences of 2-morphisms:

$$\begin{array}{ccc} \begin{array}{c} \text{G} \circ \text{H} \circ \text{H}' \circ \text{H} \circ \text{R} \\ \downarrow \text{G} \star \text{H} \star \text{H}' \star \gamma \\ \text{G} \circ \text{H} \circ \text{H}' \circ \text{S} \\ \downarrow \text{G} \star \phi \star \text{S} \\ \text{Q} \xrightarrow{\text{G} \circ \text{S}} \mathcal{P} \\ \downarrow \beta \\ \text{T} \end{array} & ; & \begin{array}{c} \text{G} \circ \text{H} \circ \text{H}' \circ \text{H} \circ \text{R} \\ \downarrow \delta \star \text{H} \star \text{H}' \star \text{R} \\ \text{F} \circ \text{H}' \circ \text{H} \circ \text{R} \\ \downarrow \text{F} \star \psi \star \text{R} \\ \text{Q} \xrightarrow{\text{F} \circ \text{R}} \mathcal{P} \\ \downarrow \alpha \\ \text{T} \end{array} \end{array}$$

We have the equality $\varepsilon = \eta$. Indeed ε precomposed with $G \star \phi^{-1} \star H \star R$ is equal to $\beta \circ (G \star \gamma)$ while η precomposed with $G \star H \star \psi^{-1} \star R$ is equal to $\alpha \circ (\delta \star R)$. The equality follows then from the equalities $\alpha \circ (\delta \star R) = \beta \circ (G \star \gamma)$ and $\phi \star H = H \star \psi$. From this one deduces that

$$\alpha \circ (F \star \gamma') = \varepsilon \circ (\delta^{-1} \star H' \star \gamma^{-1}) = \eta \circ (\delta^{-1} \star H' \star \gamma^{-1}) = \beta \circ (\delta' \star S).$$

This shows that

$$(H', \gamma', \delta') : (\mathcal{B}, S, G, \beta) \rightarrow (\mathcal{A}, R, F, \alpha)$$

is a 1-morphism in \mathfrak{D}_T . By definition of the composition in \mathfrak{D}_T ,

$$\begin{aligned} (H', \gamma', \delta') \circ (H, \gamma, \delta) &= (H' \circ H, \gamma' \circ (H' \star \gamma), \delta \circ (\delta' \star H)) \\ &= (H' \circ H, \psi \star R, G \star \psi) \end{aligned}$$

Hence ψ is a 2-morphism in \mathfrak{D}_T . Similarly ϕ is a 2-morphism in \mathfrak{D}_T . As both of them are invertible, this shows that (H', γ', δ') is a 1-morphism in \mathfrak{D}_T quasi-inverse to (H, γ, δ) . \square

2.3. Following [9, Chapitre II, §4], recall that a K -linear Abelian category \mathcal{P} is said to be finite if it is Noetherian and Artinian i.e. \mathcal{P} is essentially small and any object in \mathcal{P} has finite length. We shall say that \mathcal{P} is Hom finite if for any objects P, Q in \mathcal{P} the K -vector space $\mathcal{P}(P, Q)$ is finite dimensional. The main result of this section is the following theorem:

Theorem 2.4. — *A K -linear Abelian category \mathcal{P} which is finite and Hom finite satisfies the property **N** i.e for any quiver \mathcal{Q} and any representation $T : \mathcal{Q} \rightarrow \mathcal{P}$ the strict 2-category \mathfrak{D}_T has a 2-initial object.*

For $\mathcal{P} = \mathbf{mod}(K)$ the category of finite dimensional K -vector spaces, the result is due to M. NORI. The rest of this section will be devoted to a proof of this Theorem. The main tool is Proposition 4.1 which allows to reduce the general case to the case considered by M. NORI:

Our main motivation behind Theorem 2.4 is apply the Tannakian formalism developed by M. NORI to representations of quivers in categories of perverse sheaves.

Example 2.5. — Let k be a field of characteristic zero with an embedding $\sigma : k \hookrightarrow \mathbb{C}$ into the field \mathbb{C} of complex numbers. Given a separated k -scheme X of finite type, we consider the complex scheme X_σ obtained from X by base change along the embedding σ , and the triangulated category $D_c^b(X, \mathbb{Q})$ of complexes, with algebraically constructible cohomology, of sheaves of \mathbb{Q} -vector spaces for the classical topology on the associated analytic space $X_\sigma(\mathbb{C})$.

If A, B are objects in $D_c^b(X, \mathbb{Q})$, their Hom group is given by⁽²⁾:

$$\begin{aligned} \mathrm{Hom}(A, B) &= \mathrm{Hom}_{D_c^b(X, \mathbb{Q})}(\mathbb{Q}_X, R\mathcal{H}om(A, B)) \\ &= \mathrm{Hom}_{D^b(\mathbb{Q})}(\mathbb{Q}, R\pi_* R\mathcal{H}om(A, B)) \\ &= H^0(R\pi_* R\mathcal{H}om(A, B)) \end{aligned}$$

where $R\mathcal{H}om$ is the internal Hom and $R\pi_*$ is the direct image along the structural morphism X/k . In particular, as $R\mathcal{H}om$ and $R\pi_*$ preserve constructibility, the \mathbb{Q} -vector space $\mathrm{Hom}(A, B)$ is finite dimensional.

Let $\mathcal{P}(X) := \mathrm{Perv}(X, \mathbb{Q})$ be the heart of $D_c^b(X, \mathbb{Q})$ for the perverse t -structure. By [6, Théorème 4.3.1.(i)], the category $\mathcal{P}(X)$ is Noetherian and Artinian. As it is also Hom finite, it satisfies the assumption of Theorem 2.4.

Note that the universal property of NORI's construction is unfortunately not exactly stated in the available literature [13, 23, 7] as we have stated it in Definition 2.2. This slight variation is needed to prove Theorem 2.4 and for sake of completeness we have included the proof that NORI's category does indeed satisfy the universal property as stated in Definition 2.2 (see Theorem 3.2).

3. Recollection on Nori's construction

We denote by $\mathbf{mod}(K)$ the category of finite dimensional K -vector spaces. If V is a finite dimensional K -vector space, then the dual of V as a K -vector space is denoted by $V^\vee := \mathrm{Hom}_K(V, K)$. All K -coalgebras are assumed to be coassociative and counitary.

Let \mathcal{Q} be a quiver and

$$T : \mathcal{Q} \rightarrow \mathbf{mod}(K)$$

⁽²⁾Note that $D_c^b(\mathrm{Spec}(k), \mathbb{Q})$ is nothing but the triangulated category $D_{\mathrm{f.d.}}^b(\mathbb{Q})$ of complexes of \mathbb{Q} -vector spaces with finite dimensional cohomology, and is also equivalent to the homotopy category $K^b(\mathbf{mod}(\mathbb{Q}))$ of the Abelian category $\mathbf{mod}(\mathbb{Q})$ of finite dimensional vector spaces.

be a representation of \mathcal{Q} with values in the category of finite dimensional K -vector spaces $\mathbf{mod}(K)$. We will recall the unpublished construction of M. NORI. For detailed expositions of the work of M. NORI with complete proofs we refer to [13, 23, 7].

3.1. Assume that \mathcal{Q} has finitely many objects. Consider the ring of left endomorphisms of T . By definition, it is the subring $\text{End}_K(T)$ of

$$\prod_{q \in \mathcal{Q}} \text{End}_K(T(q))$$

formed by the elements $e = (e_q)_{q \in \mathcal{Q}}$ such that for every object $p \in \mathcal{Q}$ and every morphism $m \in \mathcal{Q}(p, q)$ the square

$$\begin{array}{ccc} T(p) & \xrightarrow{T(m)} & T(q) \\ \downarrow e_p & & \downarrow e_q \\ T(p) & \xrightarrow{T(m)} & T(q) \end{array}$$

is commutative. The ring $\text{End}_K(T)$ is a K -algebra which is finite dimensional as a K -vector space and its dual

$$A_T := \text{End}_K(T)^\vee$$

is a K -coalgebra which is finite dimensional over K . For every object $q \in \mathcal{Q}$, the finite dimensional K -vector space $T(q)$ has a natural structure of left $\text{End}_K(T)$ -module via the projection

$$\text{End}_K(T) \rightarrow \text{End}_K(T(q))$$

and thus a structure of left A_T -comodule. This shows that the representation T may be lifted to a representation

$$R_T : \mathcal{Q} \rightarrow \mathbf{comod}(A_T)$$

by simply viewing the finite dimensional vector space $T(q)$ as a left A_T -comodule. NORI's abelian category $\mathcal{A}(T)$ is then the category $\mathbf{comod}(A_T)$ of finite dimensional A_T -comodules. The representation T is then obtained as the composition of \bar{T} and the forgetful functor

$$F_T : \mathbf{comod}(A_T) \rightarrow \mathbf{mod}(K)$$

which is K -linear exact and faithful.

3.2. Let now \mathcal{Q} be a quiver which may have infinitely many objects. Consider for every finite full sub-quiver $\mathcal{Q}' \subseteq \mathcal{Q}$, the induced representation

$$T' := T|_{\mathcal{Q}'} : \mathcal{Q}' \rightarrow \mathbf{mod}(K)$$

and the associated coalgebra $A_{T'}$. If \mathcal{Q}'' is a finite full subquiver of \mathcal{Q} that contains \mathcal{Q}' , the inclusion $\mathcal{Q}' \subseteq \mathcal{Q}''$ induces by projection a morphism of K -algebras

$$\prod_{q \in \mathcal{Q}''} \text{End}_K(T(q)) \rightarrow \prod_{q \in \mathcal{Q}'} \text{End}_K(T(q))$$

which induces a morphism of K -algebras $\text{End}_K(T|_{\mathcal{Q}''}) \rightarrow \text{End}_K(T|_{\mathcal{Q}'})$. This provides a morphism of K -coalgebras $A_{T|_{\mathcal{Q}'}} \rightarrow A_{T|_{\mathcal{Q}''}}$. NORI's K -coalgebra associated with

the representation $T : \mathcal{Q} \rightarrow \mathbf{mod}(K)$ is then the coalgebra obtained by taking the colimit over all *finite full* sub-quivers of \mathcal{Q} :

$$A_T := \operatorname{colim}_{\mathcal{Q}' \subseteq \mathcal{Q}} A_{T|_{\mathcal{Q}'}}$$

and NORI's category is then the Abelian category

$$\mathcal{A}(T) := \mathbf{comod}(A_T).$$

For every object $q \in \mathcal{Q}$ the finite dimensional K -vector space $T(q)$ inherits a structure of left A_T -comodule and the representation T factors as a representation

$$R_T : \mathcal{Q} \rightarrow \mathbf{comod}(A_T) := \mathcal{A}(T)$$

via the forgetful functor $F_T : \mathbf{comod}(A_T) \rightarrow \mathbf{mod}(K)$ which is K -linear exact and faithful. Note that by construction $F_T \circ R_T = T$ and therefore $(\mathcal{A}(T), R_T, F_T, \alpha_T)$ where $\alpha_T = \operatorname{id}$ is an object in \mathfrak{D}_T .

3.3. Now let us consider the functoriality of this construction. Let $T_1 : \mathcal{Q}_1 \rightarrow \mathbf{mod}(K)$ and $T_2 : \mathcal{Q}_2 \rightarrow \mathbf{mod}(K)$ be two representations of quivers. Let $F : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$ be a morphism of quivers, and $\alpha : T_2 \circ F \rightarrow T_1$ be an invertible 2-morphism. Assume first that \mathcal{Q}_1 and \mathcal{Q}_2 have finitely many objects. Consider the morphism of rings:

$$\prod_{q_2 \in \mathcal{Q}_2} \operatorname{End}_K(T_2(q_2)) \xrightarrow{\Pi_\alpha} \prod_{q_1 \in \mathcal{Q}_1} \operatorname{End}_K(T_1(q_1)) \quad (1)$$

where the map Π_α is defined for every $e = (e_{q_2})_{q_2 \in \mathcal{Q}_2}$ by

$$\Pi_\alpha(e) = (\alpha \cdot e_{F(q_1)} \cdot \alpha^{-1})_{q_1 \in \mathcal{Q}_1}.$$

If for every object $p_2 \in \mathcal{Q}_2$ and every morphism $m_2 \in \mathcal{Q}_2(p_2, q_2)$ the square

$$\begin{array}{ccc} T_2(p_2) & \xrightarrow{T_2(m_2)} & T_2(q_2) \\ \downarrow e_{p_2} & & \downarrow e_{q_2} \\ T_2(p_2) & \xrightarrow{T_2(m_2)} & T_2(q_2) \end{array}$$

is commutative, then in particular for every object $p_1 \in \mathcal{Q}_1$ and every morphism $m_1 \in \mathcal{Q}_1(p_1, q_1)$ the square

$$\begin{array}{ccc} T_1(q_1) & \xrightarrow{T_1(m_1)} & T_1(q_1) \\ \downarrow \Pi_\alpha(e)_{p_1} & & \downarrow \Pi_\alpha(e)_{q_1} \\ T_1(p_1) & \xrightarrow{T_1(m_1)} & T_1(q_1) \end{array}$$

is commutative. This shows that the morphism of rings (1) induces a morphism of K -algebras $\operatorname{End}_K(T_2) \rightarrow \operatorname{End}_K(T_1)$ and thus a morphism of K -coalgebras: $A(F) : A_{T_1} \rightarrow A_{T_2}$. If \mathcal{Q}_1 and \mathcal{Q}_2 are not finite, then for any finite sub-quivers $\mathcal{Q}'_1 \subseteq \mathcal{Q}_1$ and $\mathcal{Q}'_2 \subseteq \mathcal{Q}_2$ such that $F(\mathcal{Q}'_1) \subseteq \mathcal{Q}'_2$ one has a morphism of K -coalgebras

$$A_{T_1|_{\mathcal{Q}'_1}} \rightarrow A_{T_2|_{\mathcal{Q}'_2}}.$$

By taking the colimit one obtains a morphism of K -coalgebras

$$A(F) : A_{T_1} \rightarrow A_{T_2}.$$

This morphism induces by restriction of scalars a K -linear exact functor

$$\mathcal{A}(F) : \mathcal{A}(T_1) := \mathbf{comod}(A_{T_1}) \rightarrow \mathcal{A}(T_2) := \mathbf{comod}(A_{T_2}).$$

The invertible 2-morphism α lifts and provides an invertible 2-morphism $\alpha : R_{T_2} \circ F \Rightarrow \mathcal{A}(F) \circ R_{T_1}$ i.e.

$$\begin{array}{ccc} \mathcal{Q}_1 & \xrightarrow{F} & \mathcal{Q}_2 \\ R_{T_1} \downarrow & \searrow \alpha & \downarrow R_{T_2} \\ \mathcal{A}(T_1) & \xrightarrow{\mathcal{A}(F)} & \mathcal{A}(T_2). \end{array}$$

3.4. Let us recall the main result related to these categories. For a complete proof we refer to [23, Satz 3.28].

Theorem 3.1 (M. Nori). — *Let \mathcal{A} be a K -linear Abelian category and $\varrho : \mathcal{A} \rightarrow \mathbf{mod}(K)$ be an exact faithful functor. Then the representation*

$$R_\varrho : \mathcal{A} \rightarrow \mathcal{A}(\varrho)$$

is a K -linear exact functor and an equivalence of categories.

As a corollary, one obtains the following result:

Theorem 3.2 (M. Nori). — *The K -linear Abelian category $\mathbf{mod}(K)$ of finite dimensional K -vector spaces satisfies the property **N**.*

Proof. — Let \mathcal{Q} be a quiver and $T : \mathcal{Q} \rightarrow \mathbf{mod}(K)$ be a representation of \mathcal{Q} in $\mathbf{mod}(K)$. The construction of M. NORI recalled in §3.2, provides a K -linear Abelian category $\mathcal{A}(T)$, a representation $R_T : \mathcal{Q} \rightarrow \mathcal{A}(T)$ and a K -linear faithful exact functor $F_T : \mathcal{A}(T) \rightarrow \mathbf{mod}(K)$ such that $F_T \circ R_T = T$. It is enough to check that the object

$$(\mathcal{A}(T), R_T, F_T, \alpha_T)$$

where $\alpha_T = \text{id}$, is 2-initial in the strict 2-category \mathfrak{D}_T i.e. satisfies the property **I**. Let \mathcal{B} be a K -linear Abelian category, $S : \mathcal{Q} \rightarrow \mathcal{B}$ be a representation, $G : \mathcal{B} \rightarrow \mathbf{mod}(K)$ be a K -linear exact faithful functor and $\beta : G \circ S \rightarrow T$ be an invertible 2-morphism.

By Theorem 3.1, the representation $R_G : \mathcal{B} \rightarrow \mathcal{A}(G)$ is a K -linear exact functor and an equivalence of categories such that

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{R_G} & \mathcal{A}(G) = \mathbf{comod}(A_G) \\ & \searrow G & \downarrow F_G \\ & & \mathbf{mod}(K) \end{array}$$

is commutative, where F_G is the forgetful functor. Note that

$$(R_G, \text{id}, \text{id}) : (\mathcal{B}, S, G, \beta) \rightarrow (\mathcal{A}(G), R_G \circ S, F_G, \beta)$$

is a 1-morphism in \mathfrak{D}_T . Since R_G is an equivalence, Lemma 2.3 assures that the objects $(\mathcal{B}, S, G, \beta)$ and $(\mathcal{A}(G), R_G \circ S, F_G, \beta)$ are then equivalent in \mathfrak{D}_T . It is therefore enough, by Remark 2.1, to prove that the property

$$\mathbf{I}((\mathcal{A}(T), R_T, F_T, \alpha_T), (\mathcal{A}(G), R_G \circ S, F_G, \beta))$$

is satisfied. By functoriality of NORI's construction as recalled in §3.3, there is a morphism of K -coalgebras $A_T \rightarrow A_G$ such that the functor

$$H : \mathcal{A}(T) := \mathbf{comod}(A_T) \rightarrow \mathbf{comod}(A_G) =: \mathcal{A}(G)$$

obtained by restriction of scalars, fits into a commutative diagram

$$\begin{array}{ccccc} \mathcal{Q} & \xrightarrow{R_T} & \mathcal{A}(T) & & \\ \downarrow S & & \downarrow H & \searrow F_T & \\ & \beta \nearrow & & & \mathbf{mod}(K) \\ & & \downarrow & \nearrow F_G & \\ \mathcal{B} & \xrightarrow{R_G} & \mathcal{A}(G) & & \end{array}$$

Hence $(H, \beta^{-1}, \text{id})$ is a 1-morphism from $(\mathcal{A}(T), R_T, F_T, \alpha_T)$ to $(\mathcal{A}(G), R_G \circ S, F_G, \beta)$. Let $H' : \mathcal{A}(T) \rightarrow \mathcal{A}(G)$ be a K -linear functor and

$$\gamma' : H' \circ R_T \xrightarrow{\sim} R_G \circ S; \quad \delta' : F_G \circ H' \xrightarrow{\sim} F_T$$

be two invertible 2-morphisms such that the square

$$\begin{array}{ccc} F_G \circ H' \circ R_T & \xrightarrow{F_G \star \gamma'} & F_G \circ R_G \circ S \\ \downarrow \delta' \star R_T & & \downarrow \beta \\ F_T \circ R_T & \xlongequal{\quad} & T \end{array} \quad (2)$$

is commutative. Let $\theta : H \rightarrow H'$ be a 2-morphism such that $\gamma' \circ (\theta \star R_T) = \beta^{-1}$ and $\delta' \circ (F_G \star \theta) = \text{id}$. Then, for every $V \in \mathbf{comod}(A_T)$, the morphism $\theta_V : H(V) \rightarrow H'(V)$ is equal to $(\delta'_V)^{-1}$ as a morphism of K -vector spaces. Hence it is unique. The existence follows from the fact that the commutativity of (2) implies that every V , of the form $V = R_T(q)$ for some $q \in \mathcal{Q}$, the morphism $(\delta'_V)^{-1}$ is a morphism of A -comodules⁽³⁾. As they generate $\mathcal{A}(T) = \mathbf{comod}(A_T)$ as an Abelian category, the morphism $\theta_V := (\delta'_V)^{-1}$ is a morphism of A -comodules for any V . The commutativity of (2) assures that $\gamma' \circ (\theta \star R_T) = \beta^{-1}$. As $\delta' \circ (F_G \star \theta) = \text{id}$ by definition, this concludes. \square

⁽³⁾The functoriality assures that β lifts as a morphism of A -comodules.

4. Finite and Hom finite K -linear Abelian categories

To obtain Theorem 2.4 from Theorem 3.2, we will need that the K -linear Abelian categories which are finite and Hom finite are precisely the categories of finite dimensional comodules over some K -coalgebra. More precisely:

Proposition 4.1. — *Let \mathcal{P} is a K -linear Abelian category. If \mathcal{P} is finite and Hom finite, then there exist a K -coalgebra A and a K -linear exact equivalence of categories between \mathcal{P} and $\text{comod}(A)$.*

Note that this characterization is simply a finite dimensional variant of the result proved by M. TAKEUCHI in [20], and that we actually deduce it from TAKEUCHI's result. As a preliminary we need the following basic lemma:

Lemma 4.2. — *Let A be K -coalgebra.*

1. *The category $\text{comod}(A)$ is the full subcategory of $\text{coMod}(A)$ formed by the Noetherian objects of $\text{coMod}(A)$.*
2. *The category $\text{comod}(A)$ is a finite and Hom finite K -linear abelian category.*

Proof. — The set of all sub-comodules of a comodule V that are finite dimensional over K is partially ordered, filtrant, and has union V (see e.g. [8, Proposition 2.3]). Hence if V is Noetherian it must be finite dimensional over K . The reciprocal follows from the fact that the forgetful functor $\text{coMod}(A) \rightarrow \text{Mod}(K)$ is K -linear exact and faithful. This implies also that every object in $\text{comod}(A)$ is Artinian. \square

Let us now prove Proposition 4.1.

Proof of Proposition 4.1. — By [9, Chapitre II, Théorème 1], we may find a locally Noetherian K -linear Abelian category \mathcal{L} and an exact⁽⁴⁾ K -linear functor $\mathcal{P} \rightarrow \mathcal{L}$ which induces an equivalence between \mathcal{P} and the full subcategory of \mathcal{L} formed by the Noetherian objects of \mathcal{L} . Since \mathcal{P} is Artinian and Hom finite, the category \mathcal{L} is locally finite and, for every objects of finite length $X, Y \in \mathcal{L}$, the K -vector space $\mathcal{L}(X, Y)$ is finite dimensional. We may thus apply [20, 5.1 Theorem, 5.8], to find a K -coalgebra A and a K -linear exact functor

$$\mathcal{L} \rightarrow \text{coMod}(A)$$

which is an equivalence between \mathcal{L} and the category $\text{coMod}(A)$ of A -comodules. The result follows then from Lemma 4.2. \square

Note that Proposition 4.1 has the following corollary:

Corollary 4.3. — *Let \mathcal{P} be a K -linear Abelian category. The following conditions are equivalent:*

1. *\mathcal{P} is finite and Hom finite;*
2. *there exist a K -coalgebra A and a K -linear exact equivalence of categories between \mathcal{P} and the category $\text{comod}(A)$;*
3. *there exists a K -linear exact and faithful functor $\omega : \mathcal{P} \rightarrow \text{mod}(K)$.*

⁽⁴⁾The exactness is proved in [9, Chapitre II, Proposition 6]

Proof. — The equivalence of conditions 1, 2 and 3 is a consequence of Lemma 4.2, Proposition 4.1 and Theorem 3.1. \square

5. Proof of Theorem 2.4

We start with the following observation:

Lemma 5.1. — *If \mathcal{P}_1 be a K -linear Abelian category that satisfies the property **N**, then every K -linear Abelian category \mathcal{P}_2 such that there exists a K -linear exact faithful functor $\omega : \mathcal{P}_2 \rightarrow \mathcal{P}_1$ satisfies also the property **N**.*

Proof. — Let $T_2 : \mathcal{Q} \rightarrow \mathcal{P}_2$ be a representation of the quiver \mathcal{Q} . We may apply the property **N** to the representation $T_1 := \omega \circ T_2 : \mathcal{Q} \rightarrow \mathcal{P}_1$, and find a K -linear Abelian category \mathcal{A}_1 , a representation $R_1 : \mathcal{Q} \rightarrow \mathcal{C}$, a faithful exact functor $F_1 : \mathcal{C}_1 \rightarrow \mathcal{P}_1$, and an invertible 2-morphism $\alpha_1 : F_1 \circ R_1 \rightarrow T_1$ such that $(\mathcal{A}, R_1, F_1, \alpha_1)$ is 2-initial in the category \mathfrak{D}_{T_1} . We set

$$R_2 := R_1 \quad \mathcal{A}_2 := \mathcal{A}_1.$$

The representation $T_2 : \mathcal{Q} \rightarrow \mathcal{P}_2$ and the K -linear exact faithful functor $\omega : \mathcal{P}_2 \rightarrow \mathcal{P}_1$, provide an object $(\mathcal{P}_2, T_2, \omega, \text{id})$ in \mathfrak{D}_{T_1} . Since $(\mathcal{A}, R_1, F_1, \alpha_1)$ is 2-initial in \mathfrak{D}_{T_1} , we obtain the existence of a K -linear functor $F_2 : \mathcal{A}_2 \rightarrow \mathcal{P}_2$ and two invertible 2-morphisms

$$\alpha_2 : F_2 \circ R_2 \xrightarrow{\sim} T_2; \quad \delta : \omega \circ F_2 \xrightarrow{\sim} F_1$$

such that the square

$$\begin{array}{ccc} \omega \circ F_2 \circ R_2 & \xrightarrow{\omega \star \alpha_2} & \omega \circ T_2 \\ \downarrow \delta \star R_2 & & \parallel \\ F_1 \circ R_1 & \xrightarrow{\alpha_1} & T_1 \end{array} \quad (3)$$

is commutative. It is enough to check that the object $(\mathcal{A}_2, R_2, F_2, \alpha_2)$ of \mathfrak{D}_{T_2} is 2-initial in \mathfrak{D}_{T_2} . For this let \mathcal{B}_2 be a K -linear Abelian category, $S_2 : \mathcal{Q} \rightarrow \mathcal{B}_2$ be a representation, $G_2 : \mathcal{B}_2 \rightarrow \mathcal{P}_2$ be a K -linear faithful exact functor, and $\beta_2 : G_2 \circ S_2 \rightarrow T_2$ be an invertible 2-morphism. Let $\mathcal{B}_1 := \mathcal{B}_2$, $S_1 := S_2$, $G_1 := \omega \circ G_2$ and $\beta_1 := \omega \star \beta_2$. Since $(\mathcal{A}_1, R_1, F_1, \alpha_1)$ is 2-initial in \mathfrak{D}_{T_1} , and $(\mathcal{B}_1, S_1, G_1, \beta_1)$ is an object in \mathfrak{D}_{T_1} , we obtain a K -linear functor $H_1 : \mathcal{A}_1 \rightarrow \mathcal{B}_1$ and two invertible 2-morphisms

$$\gamma_1 : H_1 \circ R_1 \rightarrow S_1 \quad \delta_1 : G_1 \circ H_1 \rightarrow F_1$$

such that the square

$$\begin{array}{ccc} G_1 \circ H_1 \circ R_1 = \omega \circ G_2 \circ H_1 \circ R_1 & \xrightarrow{G_1 \star \gamma_1 = \omega \star G_2 \star \gamma_1} & G_1 \circ S_1 = \omega \star G_2 \circ S_2 \\ \downarrow \delta_1 \star R_1 & & \downarrow \beta_1 = \omega \star \beta_2 \\ F_1 \circ R_1 & \xrightarrow{\alpha_1} & T_1 = \omega \circ T_2 \end{array} \quad (4)$$

is commutative. Let us set

$$H_2 := H_1 \quad \gamma_2 := \gamma_1.$$

Using the squares (3) and (5), we may apply again the fact that $(\mathcal{A}_1, R_1, F_1, \alpha_1)$ is 2-initial in \mathfrak{D}_{T_1} to the 1-morphisms of \mathfrak{D}_{T_1}

$$(G_2 \circ H_2, \beta_2 \circ (G_2 \circ \gamma_1), \delta_1) \quad (F_2, \alpha_2, \text{id})$$

to get an invertible 2-morphism $\delta_2 : G_2 \circ H_2 \rightarrow F_2$ such that $\omega \star \delta_2 = \delta_1$ and such that the square

$$\begin{array}{ccc} G_2 \circ H_2 \circ R_2 & \xrightarrow{G_2 \star \gamma_2} & G_2 \circ S_2 \\ \downarrow \delta_2 \star R_2 & & \downarrow \beta_2 \\ F_2 \circ R_2 & \xrightarrow{\alpha_2} & T_2 \end{array}$$

is commutative. Now let $H'_2 : \mathcal{A}_2 \rightarrow \mathcal{B}_2$ be a K -linear functor and

$$\gamma'_2 : H'_2 \circ R_2 \rightarrow S_2 \quad \delta'_2 : G_2 \circ H_2 \rightarrow F_2$$

be two invertible 2-morphisms such that the square

$$\begin{array}{ccc} G_2 \circ H_2 \circ R_2 & \xrightarrow{G_2 \star \gamma'_2} & G_2 \circ S_2 \\ \downarrow \delta'_2 \star R_2 & & \downarrow \beta_2 \\ F_2 \circ R_2 & \xrightarrow{\alpha_2} & T_2 \end{array} \quad (5)$$

is commutative. By composing with ω , since $\omega \star \beta_2 = \beta_1$, $\alpha_1 = \omega \star \alpha_2$, $G_1 = \omega \circ G_2$ and $F_1 = \omega \circ F_2$, we obtain the existence of a unique 2-morphism $\theta : H_2 \rightarrow H'_2$ such that

$$\gamma'_2 \circ (\theta \star R_2) = \gamma_2 \quad \omega \star (\delta'_2 \circ (G_2 \star \theta)) = \delta_1.$$

Since $\delta_1 = \omega \star \delta_2$ and ω is faithful, we obtain $\delta'_2 \circ (G_2 \star \theta) = \delta_2$ as desired. Note that θ is the unique 2-morphism $\theta : H_2 \rightarrow H'_2$ such that $\gamma'_2 \circ (\theta \star R_2) = \gamma_2$ and $\delta'_2 \circ (G_2 \star \theta) = \delta_2$ since such a 2-morphism satisfies also

$$\gamma'_2 \circ (\theta \star R_2) = \gamma_2 \quad \omega \star (\delta'_2 \circ (G_2 \star \theta)) = \omega \star \delta_2 = \delta_1.$$

This relation ensures the uniqueness since $(\mathcal{A}_1, R_1, F_1, \alpha_1)$ is 2-initial in \mathfrak{D}_{T_1} . This shows the lemma. \square

Proof of Theorem 2.4. — Let us fix a K -linear Abelian category \mathscr{P} which is finite and Hom finite. By Corollary 4.3, there exists a K -linear faithful exact functor

$$\omega : \mathscr{P} \rightarrow \text{mod}(K).$$

By applying Theorem 3.2 and Lemma 5.1, we obtain that \mathscr{P} satisfies also the property **N**. This concludes the proof. \square

6. Remarks on functoriality

In this section, we draw some simple remarks concerning the functoriality of the previous construction.

6.1. Let \mathcal{A} , \mathcal{B} and \mathcal{C} be K -linear Abelian categories and $F : \mathcal{A} \rightarrow \mathcal{C}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ be K -linear exact functors. We denote by $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ the category obtained by glueing \mathcal{A} and \mathcal{B} over \mathcal{C} . An object in this category is a 5-uplet (A, B, C, α, β) where $A \in \mathcal{A}$, $B \in \mathcal{B}$, $C \in \mathcal{C}$ and $\alpha : F(A) \rightarrow C$, $\beta : G(B) \rightarrow C$ are isomorphisms in \mathcal{C} . A morphism

$$(A, B, C, \alpha, \beta) \rightarrow (A', B', C', \alpha', \beta')$$

is a triplet (a, b, c) where $a \in \mathcal{A}(A, A')$, $b \in \mathcal{B}(B, B')$, $c \in \mathcal{C}(C, C')$ are such that $\alpha' \circ F(a) = c \circ \alpha$ and $\beta' \circ G(b) = c \circ \beta$.

This category is K -linear and Abelian with kernel and cokernel computed componentwise, and there are projection functors

$$\Pi_1 : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \rightarrow \mathcal{A} \quad \Pi_2 : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \rightarrow \mathcal{B}$$

that are K -linear and exact.

Remark 6.1. — The comparison isomorphisms $\alpha^{-1} \circ \beta$, provide a canonical isomorphism of functors $G \circ \Pi_2 \rightarrow F \circ \Pi_1$.

The following obvious remark will be useful:

Remark 6.2. — By construction if F is exact and faithful, then so is the projection functor Π_2 . Indeed assume that $\Pi_2(a, b, c) = b = 0$. Then

$$c = \beta' \circ G(b) \circ \beta^{-1} = 0$$

which implies that $F(a) = \alpha'^{-1} \circ c \circ \alpha = 0$. As F is faithful, one has also $a = 0$.

6.2. Let \mathcal{Q} be a quiver, $\mathcal{P}_1, \mathcal{P}_2$ be K -linear Abelian categories that are finite and Hom finite and

$$T_1 : \mathcal{Q} \rightarrow \mathcal{P}_1, \quad T_2 : \mathcal{Q} \rightarrow \mathcal{P}_2$$

be representation of quivers. Let $(\mathcal{A}_1, F_1, R_1, \alpha_1)$ and $(\mathcal{A}_2, F_2, R_2, \alpha_2)$ be 4-uplets obtained by applying Theorem 2.4 to the representations T_1 and T_2 respectively.

Proposition 6.3. — Let (Φ, ϕ) be a pair where $\Phi : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ is an exact K -linear functor and $\phi : \Phi \circ T_1 \rightarrow T_2$ is an isomorphism of representations. There exist an exact functor $\Psi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$, an invertible natural transformation $\rho : \Phi \circ F_1 \rightarrow F_2 \circ \Psi$, and an isomorphism of representations $\varrho : \Psi \circ R_1 \rightarrow R_2$ such that

$$\begin{array}{ccc} \Phi \circ F_1 \circ R_1 & \xrightarrow{\Phi \star \alpha_1} & \Phi \circ T_1 \\ \downarrow \rho \star R_1 & & \searrow \phi \\ F_2 \circ \Psi \circ R_1 & \xrightarrow{F_2 \star \varrho} & F_2 \circ R_2 \\ & & \nearrow \alpha_2 \\ & & T_2 \end{array}$$

is commutative.

Proof. — Consider the glued category $\mathcal{A} := \mathcal{A}_2 \times_{\mathcal{P}_2} \mathcal{P}_1$ with respect to the exact functors $F_2 : \mathcal{A}_2 \rightarrow \mathcal{P}_2$ and $\Phi : \mathcal{P}_1 \rightarrow \mathcal{P}_2$, and denote by

$$\Pi_1 : \mathcal{A} \rightarrow \mathcal{A}_2, \quad \Pi_2 : \mathcal{A} \rightarrow \mathcal{P}_1$$

the (exact) projection functors. Since F_2 is exact and faithful, so is the functor Π_2 (see Remark 6.2). Moreover we have a representation of quivers $T : \mathcal{Q} \rightarrow \mathcal{A}$:

$$\begin{aligned} p \in \mathcal{Q} &\mapsto T(p) := (R_2(p), T_1(p), T_2(p), \alpha_2(p), \phi(p)); \\ m \in \mathcal{Q}(p, q) &\mapsto T(m) := (R_2(m), T_1(m), T_2(m)). \end{aligned}$$

Note that $\Pi_2 \circ T = T_1$ and $\Pi_1 \circ T = R_2$ by definition. Apply then Theorem 2.4, with respect to the representation T_1 , to the 4-uplet $(\mathcal{A}, \Pi_2, T, \text{id})$. It yields a triplet $(\Upsilon, \gamma, \delta)$ such that

$$\begin{array}{ccc} \Pi_2 \circ \Upsilon \circ R_1 & \xrightarrow{\Pi_2 \star \gamma} & \Pi_2 \circ T \\ \downarrow \delta \star R_1 & & \parallel \\ F_1 \circ R_1 & \xrightarrow{\alpha_1} & T_1 \end{array}$$

is commutative, where $\Upsilon : \mathcal{A}_1 \rightarrow \mathcal{A}$ is an exact faithful functor, $\gamma : \Upsilon \circ R_1 \rightarrow T$ is an isomorphism of representations of quivers and $\delta : \Pi_2 \circ \Upsilon \rightarrow F_1$ is an isomorphism of functors. Now define

$$\Psi := \Pi_1 \circ \Upsilon, \quad \varrho := \Pi_1 \star \gamma.$$

There is a canonical isomorphism of functors $\varepsilon : \Phi \circ \Pi_2 \rightarrow F_2 \circ \Pi_1$, such that $\varepsilon \star T = \alpha_2^{-1} \circ \phi$. Define $\rho : \Phi \circ F_1 \rightarrow F_2 \circ \Psi$ to be the composition

$$\Phi \circ F_1 \xrightarrow{\Phi \star \delta^{-1}} \Phi \circ \Pi_2 \circ \Upsilon \xrightarrow{\varepsilon \star \Upsilon} F_2 \circ \Pi_1 \circ \Upsilon =: F_2 \circ \Psi.$$

One has then the commutative diagram

$$\begin{array}{ccccc} & \Phi \circ F_1 \circ R_1 & \xrightarrow{\Phi \star \alpha_1} & \Phi \circ T_1 & \\ & \downarrow \Phi \star \delta^{-1} \star R_1 & & \parallel & \\ & \Phi \circ \Pi_2 \circ \Upsilon \circ R_1 & \xrightarrow{(\Phi \circ \Pi_2) \star \gamma} & \Phi \circ \Pi_2 \circ T & \\ & \downarrow \varepsilon \star (\Psi \circ R_1) & & \downarrow \varepsilon \star T = \alpha_2^{-1} \circ \phi & \\ F_2 \circ \Psi \circ R_1 & \xlongequal{\quad} & F_2 \circ \Pi_1 \circ \Upsilon \circ R_1 & \xrightarrow{F_2 \star \varrho = (F_2 \circ \Pi_1) \star \gamma} & F_2 \circ \Pi_1 \circ T \end{array}$$

as desired. \square

6.3. We have the following uniqueness statement.

Proposition 6.4. — Let (Φ_1, ϕ_1) and (Φ_2, ϕ_2) be pairs where $\Phi_1, \Phi_2 : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ are exact functors and $\phi_1 : \Phi_1 \circ T_1 \rightarrow T_2$, $\phi_2 : \Phi_2 \circ T_1 \rightarrow T_2$ are isomorphisms of representations. Let $\theta : \Phi_1 \rightarrow \Phi_2$ be a natural transformation such that

$$\phi_2 \circ (\theta \star T_1) = \phi_1.$$

Let $(\Psi_1, \rho_1, \varrho_1)$, $(\Psi_2, \rho_2, \varrho_2)$ be triplets where $\Psi_1, \Psi_2 : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ are exact functors, $\rho_1 : \Phi_1 \circ F_1 \rightarrow F_2 \circ \Psi_1$, $\rho_2 : \Phi_2 \circ F_1 \rightarrow F_2 \circ \Psi_2$ are invertible natural transformations, $\varrho_1 : \Psi_1 \circ R_1 \rightarrow R_2$, $\varrho_2 : \Psi_2 \circ R_1 \rightarrow R_2$ are isomorphisms of representations such that

$$\begin{array}{ccc}
 \Phi_1 \circ F_1 \circ R_1 & \xrightarrow{\Phi_1 \star \alpha_1} & \Phi_1 \circ T_1 \\
 \downarrow \rho_1 \star R_1 & & \searrow \phi_1 \\
 & & T_2 \\
 & \nearrow \alpha_2 & \\
 F_2 \circ \Psi_1 \circ R_1 & \xrightarrow{F_2 \star \varrho_1} & F_2 \circ R_2
 \end{array}
 \quad
 \begin{array}{ccc}
 \Phi_2 \circ F_1 \circ R_1 & \xrightarrow{\Phi_2 \star \alpha_1} & \Phi_2 \circ T_1 \\
 \downarrow \rho_2 \star R_1 & & \searrow \phi_2 \\
 & & T_2 \\
 & \nearrow \alpha_2 & \\
 F_2 \circ \Psi_2 \circ R_1 & \xrightarrow{F_2 \star \varrho_2} & F_2 \circ R_2
 \end{array}$$

are commutative. Then there exists one and only natural transformation $\vartheta : \Psi_1 \rightarrow \Psi_2$ such that

$$\begin{array}{ccc}
 \Phi_1 \circ F_1 & \xrightarrow{\rho_1} & F_2 \circ \Psi_1 \\
 \downarrow \theta \star F_1 & & \downarrow F_2 \star \vartheta \\
 \Phi_2 \circ F_1 & \xrightarrow{\rho_2} & F_2 \circ \Psi_2
 \end{array}
 \quad
 \begin{array}{ccc}
 \Psi_1 \circ R_1 & & \\
 \downarrow \vartheta \circ R_1 & \searrow \varrho_1 & \\
 \Psi_2 \circ R_1 & \xrightarrow{\varrho_2} & R_2
 \end{array}$$

are commutative.

6.4. There is also a variant of Proposition 6.3 where one starts with two quivers \mathcal{Q}_1 , \mathcal{Q}_2 , a morphism of quivers $Q : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$ and two representations

$$T_1 : \mathcal{Q}_1 \rightarrow \mathcal{P}_1, \quad T_2 : \mathcal{Q}_2 \rightarrow \mathcal{P}_2.$$

Let $(\mathcal{A}_1, F_1, R_1, \alpha_1)$ and $(\mathcal{A}_2, F_2, R_2, \alpha_2)$ be 4-uplets obtained by applying Theorem 2.4 to the representations T_1 and T_2 respectively.

Proposition 6.5. — Let (Φ, ϕ) be a pair where $\Phi : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ is an exact K -linear functor and $\phi : \Phi \circ T_1 \rightarrow T_2 \circ Q$ is an isomorphism of representations. There exist an exact functor $\Psi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$, an invertible natural transformation $\rho : \Phi \circ F_1 \rightarrow F_2 \circ \Psi$, and an isomorphism of representations $\varrho : \Psi \circ R_1 \rightarrow R_2 \circ Q$ such that

$$\begin{array}{ccc}
 \Phi \circ F_1 \circ R_1 & \xrightarrow{\Phi \star \alpha_1} & \Phi \circ T_1 \\
 \downarrow \rho \star R_1 & & \searrow \phi \\
 & & T_2 \circ Q \\
 & \nearrow \alpha_2 \star Q & \\
 F_2 \circ \Psi \circ R_1 & \xrightarrow{F_2 \star \varrho} & F_2 \circ R_2 \circ Q
 \end{array} \tag{6}$$

is commutative.

Proof. — The proof reduces to Proposition 6.3. Indeed consider the representations $\mathcal{Q}_1 : T_1 : \mathcal{Q}_1 \rightarrow \mathcal{P}_1$ and $T'_2 := T_2 \circ Q : \mathcal{Q}_1 \rightarrow \mathcal{P}_2$ of the quiver \mathcal{Q}_1 . Let $(\mathcal{A}'_2, F'_2, R'_2, \alpha'_2)$

be a 4-uplet obtained by applying Theorem 2.4 to the representations T'_2 . By Proposition 6.3 applied to the pair (Φ, ϕ) , there exist an exact functor $\Psi' : \mathcal{A}_1 \rightarrow \mathcal{A}'_2$, an invertible natural transformation $\rho' : \Phi \circ F_1 \rightarrow F'_2 \circ \Psi'$, and an isomorphism of representations $\varrho' : \Psi' \circ R_1 \rightarrow R'_2$ such that

$$\begin{array}{ccc}
 \Phi \circ F_1 \circ R_1 & \xrightarrow{\Phi \star \alpha_1} & \Phi \circ T_1 \\
 \downarrow \rho' \star R_1 & & \searrow \phi \\
 F'_2 \circ \Psi' \circ R_1 & \xrightarrow{F'_2 \star \varrho'} & F'_2 \circ R'_2 \\
 & & \nearrow \alpha'_2 \\
 & & T'_2
 \end{array} \tag{7}$$

is commutative. Now apply Theorem 2.4, with respect to T'_2 , to the 4-uplet $(\mathcal{A}'_2, F_2, R_2 \circ Q, \alpha_2)$. We obtain a triplet $(\Upsilon, \gamma, \delta)$ such that

$$\begin{array}{ccc}
 F_2 \circ \Upsilon \circ R'_2 & \xrightarrow{F_2 \star \gamma} & F_2 \circ R_2 \circ Q \\
 \downarrow \delta \star R'_2 & & \downarrow \alpha_2 \\
 F'_2 \circ R'_2 & \xrightarrow{\alpha'_2} & T'_2 := T_2 \circ Q
 \end{array} \tag{8}$$

is commutative, where $\Upsilon : \mathcal{A}'_2 \rightarrow \mathcal{A}_2$ is an exact faithful functor, $\gamma : \Upsilon \circ R'_2 \rightarrow R_2 \circ Q$ is an isomorphism of representations of quivers and $\delta : F_2 \circ \Upsilon \rightarrow F'_2$ is an isomorphism of functors. Now let $\Psi = \Upsilon \circ \Psi'$, define the natural transformation $\rho : \Phi \circ F_1 \rightarrow F_2 \circ \Psi$ as the composition:

$$\Phi \circ F_1 \xrightarrow{\rho'} F'_2 \circ \Psi' \xrightarrow{\delta^{-1} \star \Psi'} F_2 \circ \Upsilon \circ \Psi' = F_2 \circ \Psi$$

and $\varrho : \Psi \circ R_1 \rightarrow R_2 \circ Q$ as the composition

$$\Psi \circ R_1 = \Upsilon \circ \Psi' \circ R_1 \xrightarrow{\Upsilon \star \varrho'} \Upsilon \circ R'_2 \xrightarrow{\gamma} R_2 \circ Q.$$

The commutativity of (6) is then an immediate consequence of the commutativity of the diagrams (7) and (8). \square

The uniqueness statement in Proposition 6.4 admits then the following variant:

Proposition 6.6. — *Let (Φ_1, ϕ_1) and (Φ_2, ϕ_2) be pairs where $\Phi_1, \Phi_2 : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ are exact functors and $\phi_1 : \Phi_1 \circ T_1 \rightarrow T_2 \circ Q$, $\phi_2 : \Phi_2 \circ T_1 \rightarrow T_2 \circ Q$ are isomorphisms of representations. Let $\theta : \Phi_1 \rightarrow \Phi_2$ be a natural transformation such that*

$$\phi_2 \circ (\theta \star T_1) = \phi_1.$$

Let $(\Psi_1, \rho_1, \varrho_1)$, $(\Psi_2, \rho_2, \varrho_2)$ be triplets where $\Psi_1, \Psi_2 : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ are exact functors, $\rho_1 : \Phi_1 \circ F_1 \rightarrow F_2 \circ \Psi_1$, $\rho_2 : \Phi_2 \circ F_1 \rightarrow F_2 \circ \Psi_2$ are invertible natural transformations,

$\varrho_1 : \Psi_1 \circ R_1 \rightarrow R_2 \circ Q$, $\varrho_2 : \Psi_2 \circ R_1 \rightarrow R_2 \circ Q$ are isomorphisms of representations such that

$$\begin{array}{ccc}
 \Phi_1 \circ F_1 \circ R_1 & \xrightarrow{\Phi_1 \star \alpha_1} & \Phi_1 \circ T_1 \\
 \downarrow \rho_1 \star R_1 & & \searrow \phi_1 \\
 & & T_2 \circ Q \\
 & \nearrow \alpha_2 \star Q & \\
 F_2 \circ \Psi_1 \circ R_1 & \xrightarrow{F_2 \star \varrho_1} & F_2 \circ R_2 \circ Q
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Phi_2 \circ F_1 \circ R_1 & \xrightarrow{\Phi_2 \star \alpha_1} & \Phi_2 \circ T_1 \\
 \downarrow \rho_2 \star R_1 & & \searrow \phi_2 \\
 & & T_2 \circ Q \\
 & \nearrow \alpha_2 \star Q & \\
 F_2 \circ \Psi_2 \circ R_1 & \xrightarrow{F_2 \star \varrho_2} & F_2 \circ R_2 \circ Q
 \end{array}$$

are commutative. Then there exists one and only natural transformation $\vartheta : \Psi_1 \rightarrow \Psi_2$ such that

$$\begin{array}{ccc}
 \Phi_1 \circ F_1 & \xrightarrow{\rho_1} & F_2 \circ \Psi_1 \\
 \theta \star F_1 \downarrow & & \downarrow F_2 \star \vartheta \\
 \Phi_2 \circ F_1 & \xrightarrow{\rho_2} & F_2 \circ \Psi_2
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Psi_1 \circ R_1 & & \\
 \vartheta \circ R_1 \downarrow & \searrow \varrho_1 & \\
 \Psi_2 \circ R_1 & \xrightarrow{\varrho_2} & R_2 \circ Q
 \end{array}$$

are commutative.

6.5. Consider two finite and Hom finite K -linear Abelian categories \mathcal{P} and \mathcal{Q} . There is finally a useful consequence of Proposition 6.5 where one starts with a quiver \mathcal{Q} , a representation of quivers

$$T : \mathcal{Q} \rightarrow \mathcal{P}$$

a K -linear Abelian category \mathcal{B} , a representation $S : \mathcal{Q} \rightarrow \mathcal{B}$ and a faithful exact functor $G : \mathcal{B} \rightarrow \mathcal{Q}$. We consider a 4-uplet $(\mathcal{A}, F, R, \alpha)$ obtained by applying Theorem 2.4 to the representation T .

Proposition 6.7. — Let (Φ, ϕ) be a pair where $\Phi : \mathcal{P} \rightarrow \mathcal{Q}$ is an exact K -linear functor and $\phi : \Phi \circ T \rightarrow G \circ S$ is an isomorphism of representations. There exist an exact functor $\Psi : \mathcal{A} \rightarrow \mathcal{B}$, an invertible natural transformation $\rho : \Phi \circ F \rightarrow G \circ \Psi$, and an isomorphism of representations $\varrho : \Psi \circ R \rightarrow S$ such that

$$\begin{array}{ccc}
 \Phi \circ F \circ R & \xrightarrow{\Phi \star \alpha} & \Phi \circ T \\
 \rho \star R \downarrow & & \downarrow \phi \\
 G \circ \Psi \circ R & \xrightarrow{G \star \varrho} & G \circ S
 \end{array}$$

is commutative.

Proof. — Let $\mathcal{Q}_1 := \mathcal{Q}$, $\mathcal{P}_1 := \mathcal{P}$ and $T_1 := T$. Now consider the category \mathcal{B} as a quiver $\mathcal{Q}_2 := \mathcal{B}$, and set $\mathcal{P}_2 := \mathcal{Q}$, $T_2 := G$ and $Q := S$. The 4-uplet $(\mathcal{B}, G, \text{id}, \text{id})$ satisfies the universal property for the representation T_2 . Apply Proposition 6.5. \square

The uniqueness statement in Proposition 6.6 implies then the following:

Proposition 6.8. — Let (Φ_1, ϕ_1) and (Φ_2, ϕ_2) be pairs where $\Phi_1, \Phi_2 : \mathcal{P} \rightarrow \mathcal{Q}$ are exact functors and $\phi_1 : \Phi_1 \circ T \rightarrow G \circ S$, $\phi_2 : \Phi_2 \circ T \rightarrow G \circ S$ are isomorphisms of representations. Let $\theta : \Phi_1 \rightarrow \Phi_2$ be a natural transformation such that

$$\phi_2 \circ (\theta \star T) = \phi_1.$$

Let $(\Psi_1, \rho_1, \varrho_1)$, $(\Psi_2, \rho_2, \varrho_2)$ be triplets where $\Psi_1, \Psi_2 : \mathcal{A} \rightarrow \mathcal{B}$ are exact functors, $\Phi_1 \circ F \rightarrow G \circ \Psi_1$, $\rho_2 : \Phi_2 \circ F \rightarrow G \circ \Psi_2$ are invertible natural transformations, $\varrho_1 : \Psi_1 \circ R \rightarrow S$, $\varrho_2 : \Psi_2 \circ R \rightarrow S$ are isomorphisms of representations such that

$$\begin{array}{ccc} \Phi_1 \circ F \circ R & \xrightarrow{\Phi_1 \star \alpha} & \Phi_1 \circ T \\ \rho_1 \star R \downarrow & & \downarrow \phi_1 \\ G \circ \Psi_1 \circ R & \xrightarrow{G \star \varrho_1} & G \circ S \end{array} \quad \begin{array}{ccc} \Phi_2 \circ F \circ R & \xrightarrow{\Phi_2 \star \alpha} & \Phi_2 \circ T \\ \rho_2 \star R \downarrow & & \downarrow \phi_2 \\ G \circ \Psi_2 \circ R & \xrightarrow{G \star \varrho_2} & G \circ S \end{array}$$

are commutative. Then there exists one and only natural transformation $\vartheta : \Psi_1 \rightarrow \Psi_2$ such that

$$\begin{array}{ccc} \Phi_1 \circ F & \xrightarrow{\rho_1} & G \circ \Psi_1 \\ \theta \star F \downarrow & & \downarrow F_2 \star \vartheta \\ \Phi_2 \circ F & \xrightarrow{\rho_2} & G \circ \Psi_2 \end{array} \quad \begin{array}{ccc} \Psi_1 \circ R & & \\ \vartheta \circ R \downarrow & \searrow \varrho_1 & \\ \Psi_2 \circ R & \xrightarrow{\varrho_2} & G \circ S \end{array}$$

are commutative.

7. Abelian categories of Perverse motives

Recall that $k = \mathbb{C}$ is the field of complex numbers (one can also choose a field of characteristic zero k with a fixed embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$) and that by a k -variety we mean a quasi-projective k -scheme.

In this section we use Theorem 2.4 to construct a perverse analog over quasi-projective k -schemes of the category $\text{NMM}(k, \mathbb{Q})$.

7.1. Let X be a k -variety. We denote by $\mathcal{M}(X)$ one of the following Abelian categories: (a) the category of perverse sheaves $\mathcal{P}(X)$; (b) the category of perverse sheaves of geometric origins $\mathcal{P}(X)^{\text{go}}$ introduced in [6, 6.2.4]; (c) the category of mixed Hodge modules $\text{MHM}(X, \mathbb{Q})$; (d) the category $\text{MHM}(X, \mathbb{Q})^{\text{go}}$ of mixed Hodge modules of geometric origins (see [19, (2.6) Définition]). Recall that the derived categories $D^b(\mathcal{M}(X))$, as X runs over k -varieties, are endowed with a six functors formalism

$$D^b(\mathcal{M}(X)) \begin{array}{c} \xleftarrow{f^*_{\mathcal{M}}} \\ \xrightarrow{f_*^{\mathcal{M}}} \end{array} D^b(\mathcal{M}(Y)) \begin{array}{c} \xrightarrow{f_!^{\mathcal{M}}} \\ \xleftarrow{f^!_{\mathcal{M}}} \end{array} D^b(\mathcal{M}(X)).$$

We denote by

$$H^i_{\mathcal{M}} : D^b(\mathcal{M}(X)) \rightarrow \mathcal{M}(X) \quad i \in \mathbb{Z}$$

the cohomological functor associated with the usual t -structure.

7.2. A relative triplet is a triplet (Y, Z, i) where Y is a k -variety with a morphism of k -varieties $a : Y \rightarrow X$, Z is a closed subscheme of Y and $i \in \mathbb{Z}$ is an integer.

Definition 7.1. — Let (Y, Z, i) be a relative X -triplet. We set

$$\mathbb{T}_X^{\mathcal{M}}(Y, Z, i) := \mathrm{H}_{\mathcal{M}}^{-i}(a_!^{\mathcal{M}}(u_*^{\mathcal{M}}u^!a^!_{\mathcal{M}}(\mathbb{Q}_X^{\mathcal{M}})))$$

where $u : U \hookrightarrow Y$ is the open immersion of the complement of Z in Y .

Note that by definition $\mathbb{T}_X^{\mathcal{M}}(Y, Z, i)$ is an object in $\mathcal{M}(X)$ which depends only on the reduced structure of Y and Z . Using the formalism of the six operations, it is easy to construct to different sorts of morphisms between these objects in $\mathcal{M}(X)$: the functoriality morphisms and the boundary morphisms.

Let (Y_1, Z_1, i) and (Y_2, Z_2, i) be relative X -triplets. Assume that $f : Y_2 \rightarrow Y_1$ is a morphism of X -schemes, such that $f(Z_2) \subseteq Z_1$. The functoriality morphisms are maps in $\mathcal{M}(X)$

$$f_{\star}^{\mathcal{M}} : \mathbb{T}_X^{\mathcal{M}}(Y_2, Z_2, i) \rightarrow \mathbb{T}_X^{\mathcal{M}}(Y_1, Z_1, i) \quad (9)$$

such that if (Y_3, Z_3, i) is a relative X -triplet, and $g : Y_3 \rightarrow Y_2$ is a morphism of X -schemes such that $g(Z_3) \subseteq Z_2$, then

$$f_{\star}^{\mathcal{M}} \circ g_{\star}^{\mathcal{M}} = (fg)_{\star}^{\mathcal{M}}.$$

Let (Y, Z, i) be a relative X -triplet, and $W \subseteq Z$ be a closed subset. Then the boundary morphism is a map in $\mathcal{M}(X)$

$$\mathbb{T}_X^{\mathcal{M}}(Y, Z, i) \rightarrow \mathbb{T}_X^{\mathcal{M}}(Z, W, i - 1). \quad (10)$$

7.3. Let $\mathcal{D}_X^{\mathrm{Nori}}$ be the quiver with a vertex for every relative X -triplet (Y, Z, i) and the following edges.

- Let (Y_1, Z_1, i) and (Y_2, Z_2, i) be relative X -triplets. Then every morphism of X -schemes $f : Y_2 \rightarrow Y_1$ such that $f(Z_2) \subseteq Z_1$ defines an edge

$$(Y_2, Z_2, i) \rightarrow (Y_1, Z_1, i) \quad (11)$$

- For every relative X -triplet (Y, Z, i) and every closed subscheme $W \subseteq Z$, we have an edge

$$(Y, Z, i) \rightarrow (Z, W, i - 1). \quad (12)$$

The quiver $\mathcal{D}_X^{\mathrm{Nori}}$ admits then a representation

$$\mathbb{T}_X^{\mathcal{M}} : \mathcal{D}_X^{\mathrm{Nori}} \rightarrow \mathcal{M}(X)$$

where the edges (11), and (12) are maps respectively to the morphisms (9) and (10). As the functor $\mathrm{rat}_X^{\mathcal{H}} : \mathrm{MHM}(X, \mathbb{Q}) \rightarrow \mathcal{P}(X)$ is compatible with the formalism of cohomological operations, there is a canonical isomorphism of representations of quivers

$$\alpha_X^{\mathcal{H}} : \mathrm{rat}_X^{\mathcal{H}} \circ \mathbb{T}_X^{\mathcal{H}} \rightarrow \mathbb{T}_X^{\mathcal{P}}$$

7.4. Let X be a k -variety. Since the Abelian category of perverse sheaves $\mathcal{P}(X)$ is Noetherian, Artinian and has finite dimensional Homs (see Example 2.5), we may apply Theorem 2.4 , to the representation

$$\mathbf{T}_X^{\mathcal{P}} : \mathcal{D}_X^{\text{Nori}} \rightarrow \mathcal{P}(X)$$

to obtain an Abelian category $\mathcal{N}^{\text{eff}}(X)$ together with a faithful exact functor $\text{rat}_X^{\mathcal{N}}$, a representation $\mathbf{T}_X^{\mathcal{N}}$:

$$\text{rat}_X^{\mathcal{N}} : \mathcal{N}^{\text{eff}}(X) \rightarrow \mathcal{P}(X) \quad \mathbf{T}_X^{\mathcal{N}} : \mathcal{D}_X^{\text{Nori}} \rightarrow \mathcal{N}^{\text{eff}}(X)$$

and an isomorphism of representations of quivers $\alpha_X^{\text{Nori}} : \text{rat}_X^{\mathcal{N}} \circ \mathbf{T}_X^{\mathcal{N}} \rightarrow \mathbf{T}_X^{\mathcal{P}}$ such that the following theorem holds:

Theorem 7.2. — *For every k -variety X , the 4-uplet*

$$(\mathcal{N}^{\text{eff}}(X), \text{rat}_X^{\mathcal{N}}, \mathbf{T}_X^{\mathcal{N}}, \alpha_X^{\mathcal{N}})$$

has the following properties.

- *For every $(\mathcal{A}, \mathbf{B}, \mathbf{T}, \alpha)$ where $\mathbf{B} : \mathcal{A} \rightarrow \mathcal{P}(X)$ is a faithful exact functor, $\mathbf{T} : \mathcal{D}_X^{\text{Nori}} \rightarrow \mathcal{A}$ is a representation, and $\alpha : \mathbf{B} \circ \mathbf{T} \rightarrow \mathbf{T}_X^{\mathcal{P}}$ is an isomorphism of representations of quivers, there exists a triplet $(\mathbf{R}, \rho, \varrho)$ where*

$$\mathbf{R} : \mathcal{N}^{\text{eff}}(X) \rightarrow \mathcal{A}$$

is a faithful exact functor, $\rho : \mathbf{B} \circ \mathbf{R} \rightarrow \text{rat}_X^{\mathcal{N}}$ is an invertible natural transformation and $\varrho : \mathbf{R} \circ \mathbf{T}_X^{\mathcal{N}} \rightarrow \mathbf{T}$ is an isomorphism of quiver representations such that

$$\begin{array}{ccc} \mathbf{B} \circ \mathbf{R} \circ \mathbf{T}_X^{\mathcal{N}} & \xrightarrow{\mathbf{B} \circ \varrho} & \mathbf{B} \circ \mathbf{T} \\ \downarrow \rho \circ \mathbf{T}_X^{\mathcal{N}} & & \downarrow \alpha \\ \text{rat}_X^{\mathcal{N}} \circ \mathbf{T}_X^{\mathcal{N}} & \xrightarrow{\alpha_X^{\mathcal{N}}} & \mathbf{T}_X^{\mathcal{P}} \end{array}$$

is commutative.

- *If $(\mathbf{R}', \varrho', \rho')$ is another such triplet, then there exists one and only one natural transformation $\theta : \mathbf{R} \rightarrow \mathbf{R}'$ such that*

$$\rho' \circ (\mathbf{B} \circ \theta) = \rho \quad \varrho' \circ (\theta \circ \mathbf{T}_X^{\mathcal{N}}) = \varrho.$$

Remark 7.3. — The category $\mathcal{N}^{\text{eff}}(\text{Spec}(k))$ is nothing but the category of effective (homological) motives $\text{EHM}(k)$ constructed by M. NORI (see [16] or [13, Definition 3.13]).

By applying Theorem 7.2 to the 4-uplet

$$(\text{MHM}(X, \mathbb{Q}), \text{rat}_X^{\mathcal{H}}, \mathbf{T}_X^{\mathcal{H}}, \alpha_X^{\mathcal{H}})$$

where $\text{rat}_X^{\mathcal{H}} : \text{MHM}(X, \mathbb{Q}) \rightarrow \mathcal{P}(X)$ is the forgetful functor, one obtains a faithful exact functor

$$\mathbf{R}_X^{\mathcal{H}} : \mathcal{N}^{\text{eff}}(X) \rightarrow \text{MHM}(X, \mathbb{Q})$$

which takes its values in the full Abelian subcategory formed by the mixed Hodge modules of geometric origin.

7.5. We now want to apply Proposition 6.5 to construct the categories of perverse motives out of the categories of effective perverse motives.

Let \mathcal{A} be a K -linear Abelian category and $\mathbf{L} : \mathcal{A} \rightarrow \mathcal{A}$ be a K -linear exact functor. Let us construct a category $\mathcal{A}[\mathbf{L}^{-1}]$ as follows. An object in $\mathcal{A}[\mathbf{L}^{-1}]$ is a pair (A, n) where A is an object in \mathcal{A} and $n \in \mathbb{Z}$ is an integer. Morphisms in $\mathcal{A}[\mathbf{L}^{-1}]$ are given by

$$\mathrm{Hom}_{\mathcal{A}[\mathbf{L}^{-1}]}((A, n), (B, m)) = \mathrm{colim}_i \mathrm{Hom}_{\mathcal{A}}(\mathbf{L}^{\circ i+n}(A), \mathbf{L}^{\circ i+m}(B)).$$

where the colimit is taken over all integers $i \in \mathbb{N}$ such that $i+n \geq 0$ and $i+m \geq 0$. There is a functor

$$\begin{aligned} \mathcal{A} &\rightarrow \mathcal{A}[\mathbf{L}^{-1}] \\ A &\mapsto (A, 0). \end{aligned}$$

Lemma 7.4. — *The category $\mathcal{A}[\mathbf{L}^{-1}]$ is K -linear Abelian and the functor $\mathcal{A} \rightarrow \mathcal{A}[\mathbf{L}^{-1}]$ is K -linear and exact. If \mathbf{L} is faithful (resp. full) then $\mathcal{A} \rightarrow \mathcal{A}[\mathbf{L}^{-1}]$ is a faithful (resp. full) functor.*

Proof. — We only sketch the proof as the details are straightforward. The category $\mathcal{A}[\mathbf{L}^{-1}]$ is K -linear and admits finite direct sums given by

$$(A, n) \oplus (B, m) = (\mathbf{L}^{i+n}(A) \oplus \mathbf{L}^{\circ i+m}(B), -i)$$

where $i \in \mathbb{N}$ is some integer such that $i+n \geq 0$ and $i+m \geq 0$. Let $\alpha : (A, n) \rightarrow (B, m)$ be a morphism in $\mathcal{A}[\mathbf{L}^{-1}]$. Let $i \in \mathbb{N}$ an integer such that $i+n \geq 0$ and $i+m \geq 0$ and $a : \mathbf{L}^{\circ i+n}(A) \rightarrow \mathbf{L}^{\circ i+m}(B)$ a morphism in \mathcal{A} that lifts α . The maps $\mathrm{Ker} a \rightarrow \mathbf{L}^{\circ i+n}$ and $\mathbf{L}^{i+m}(B) \rightarrow \mathrm{Coker} a$ in \mathcal{A} define maps

$$(\mathrm{Ker} a, -i) \rightarrow (A, n) \quad (B, m) \rightarrow (\mathrm{Coker} a, -i)$$

in $\mathcal{A}[\mathbf{L}^{-1}]$. Using that \mathbf{L} is exact, it is easy to check that these maps are respectively a kernel and a cokernel of α in $\mathcal{A}[\mathbf{L}^{-1}]$. Since \mathcal{A} is an Abelian category, the canonical map

$$(A, n) / \mathrm{Ker} \alpha = (\mathbf{L}^{\circ i+n}(A) / \mathrm{Ker} a, -i) \rightarrow (\mathrm{Im} a, -i) = \mathrm{Im} \alpha$$

is an isomorphism (here $\mathrm{Im} \alpha$ denotes, as usual, the kernel of the cokernel map). \square

Remark 7.5. — Let $A \in \mathcal{A}$, and $n \in \mathbb{Z}$, $r \in \mathbb{N}$ be integers. For every integer $i \in \mathbb{N}$ such that $i+n-r \geq 0$, the identity of $\mathbf{L}^{\circ i+n}(A) = \mathbf{L}^{\circ i+n-r}(\mathbf{L}^{\circ r}(A))$ induces an isomorphism between (A, n) and $(\mathbf{L}^{\circ r}(A), n-r)$ in $\mathcal{A}[\mathbf{L}^{-1}]$. In particular (A, r) is isomorphic to $(\mathbf{L}^{\circ r}(A), 0)$ in $\mathcal{A}[\mathbf{L}^{-1}]$.

Recall that by definition, for $\mathcal{M} \in \{\mathcal{H}, \mathcal{P}\}$

$$\mathbb{Q}_k^{\mathcal{M}}(1) = \mathbf{T}_k^{\mathcal{M}}(\mathbf{G}_{m,k}, \{1\}, 1).$$

The Tate twist of an object $A \in \mathcal{M}(X)$ is then defined by $A(1) := A \boxtimes \mathbb{Q}_k^{\mathcal{M}}(1)$ where $\boxtimes : \mathcal{M}(X) \times \mathcal{M}(k) \rightarrow \mathcal{M}(X)$ is the external product.

Let X be a quasi-projective k -scheme. Consider the quivers $\mathcal{Q}_1 := \mathcal{Q}_2 = \mathcal{D}_X^{\mathrm{Nori}}$ and the representations

$$\mathbf{T}_1 := \mathbf{T}_2 := \mathbf{T}_X^{\mathcal{M}}.$$

Consider the morphism of quivers $Q : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$.

$$(Y, Z, i) \mapsto (\mathbf{G}_{m,Y}, \mathbf{G}_{m,Z} \cup Y, i + 1)$$

where Y is embedded in $\mathbf{G}_{m,Y}$ via the unit section. Denote by $\Phi_X^{\mathcal{M}} : \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ the functor which maps K to its twist $K(1)$.

Lemma 7.6. — *There are canonical isomorphisms of representations of quivers*

$$\phi_X^{\mathcal{M}} : \Phi_X^{\mathcal{M}} \circ \mathbf{T}_X^{\mathcal{M}} \rightarrow \mathbf{T}_X^{\mathcal{M}} \circ Q$$

which are compatible via the functor $\mathbf{rat}_X^{\mathcal{H}}$.

Proof. — Let $z : Z \hookrightarrow Y$ be a closed immersion, and $u : U \hookrightarrow Y$ be its open complement. We denote by $\pi : \mathbf{G}_{m,k} \rightarrow \mathrm{Spec}(k)$ the projection and $v : \mathbf{G}_{m,k} \setminus \{1\} \hookrightarrow \mathbf{G}_{m,k}$ the open immersion. The open immersion in $\mathbf{G}_{m,Y}$ of the complement of $\mathbf{G}_{m,Z} \cup Y$ is

$$u \times_k v : U \times_k (\mathbf{G}_{m,k} \setminus \{1\}) \hookrightarrow \mathbf{G}_{m,Y}.$$

Let $\sigma : \mathrm{Spec}(k) \rightarrow \mathbf{G}_{m,k}$ be the unit section. The triangle $\sigma_!^{\mathcal{M}} \sigma^!_{\mathcal{M}} \rightarrow \mathrm{Id} \rightarrow v_*^{\mathcal{M}} v^!_{\mathcal{M}} \xrightarrow{+1}$ yields the distinguished triangle

$$\mathbb{Q}_k^{\mathcal{M}} \rightarrow \pi_!^{\mathcal{M}} \pi^!_{\mathcal{M}}(\mathbb{Q}_k^{\mathcal{M}}) \rightarrow \pi_!^{\mathcal{M}} v_*^{\mathcal{M}} v^!_{\mathcal{M}} \pi^!_{\mathcal{M}}(\mathbb{Q}_k^{\mathcal{M}}) \xrightarrow{+1}.$$

Via the canonical isomorphism $\pi_!^{\mathcal{M}} \pi^!_{\mathcal{M}}(\mathbb{Q}_k^{\mathcal{M}}) = \mathbb{Q}_k^{\mathcal{M}} \oplus \mathbb{Q}_k^{\mathcal{M}}(1)[1]$, one gets

$$\pi_!^{\mathcal{M}} v_*^{\mathcal{M}} v^!_{\mathcal{M}} \pi^!_{\mathcal{M}}(\mathbb{Q}_k^{\mathcal{M}}) = \mathbb{Q}_k^{\mathcal{M}}(1)[1].$$

We have thus an isomorphism

$$(u \times_k v)_!^{\mathcal{M}} (a \times_k \pi)_!^{\mathcal{M}}(\mathbb{Q}_X^{\mathcal{M}}) = u_!^{\mathcal{M}} a_!^{\mathcal{M}}(\mathbb{Q}_X^{\mathcal{M}}) \boxtimes v^! \pi^!_{\mathcal{M}}(\mathbb{Q}_k^{\mathcal{M}}).$$

The object $(a \times_k \pi)_!^{\mathcal{M}} (u \times_k v)_*^{\mathcal{M}} (u \times_k v)_!^{\mathcal{M}} (a \times_k \pi)_!^{\mathcal{M}}(\mathbb{Q}_X^{\mathcal{M}})$ of $D^b(\mathcal{P}(X))$ is therefore isomorphic to

$$(a_!^{\mathcal{M}} u_*^{\mathcal{M}} u^!_{\mathcal{M}} a_!^{\mathcal{M}}(\mathbb{Q}_X^{\mathcal{M}})) \boxtimes (\pi_!^{\mathcal{M}} v_*^{\mathcal{M}} v^!_{\mathcal{M}} \pi^!_{\mathcal{M}}(\mathbb{Q}_k^{\mathcal{M}})) = a_!^{\mathcal{M}} u_*^{\mathcal{M}} u^!_{\mathcal{M}} a_!^{\mathcal{M}}(\mathbb{Q}_X^{\mathcal{M}})(1)[1].$$

Hence for every integer $i \in \mathbb{Z}$, we have a canonical isomorphism

$$\mathbf{T}_X^{\mathcal{M}}(\mathbf{G}_{m,Y}, \mathbf{G}_{m,Z} \cup Y, i + 1) = \mathbf{T}_X^{\mathcal{M}}(Y, Z, i)(1).$$

The right hand side is $\mathbf{T}_X^{\mathcal{M}}(Q(Y, Z, i))$ while the right hand side is $\Phi_X^{\mathcal{M}}(\mathbf{T}_X^{\mathcal{M}}(X, Z, i))$. One checks easily that this defines an isomorphism of quivers as desired and this concludes the proof. \square

Note that $\Phi := \Phi_X^{\mathcal{P}}$ is an exact functor. Hence we may apply Proposition 6.5 which yields an exact functor $L : \mathcal{N}^{\mathrm{eff}}(X) \rightarrow \mathcal{N}^{\mathrm{eff}}(X)$, an invertible natural transformation $\rho : \Phi \circ \mathbf{rat}_X^{\mathcal{N}} \rightarrow \mathbf{rat}_X^{\mathcal{N}} \circ L$, and an isomorphism of representations of quivers $\varrho : L \circ \mathbf{T}_X^{\mathcal{N}} \rightarrow$

$T_X^{\mathcal{N}} \circ Q$ such that

$$\begin{array}{ccc}
 \Phi \circ \text{rat}_X^{\mathcal{N}} \circ T_X^{\mathcal{N}} & \xrightarrow{\Phi \star \alpha_X^{\mathcal{N}}} & \Phi \circ T_X^{\mathcal{N}} \\
 \downarrow \rho \star T_X^{\mathcal{N}} & & \searrow \phi_X^{\mathcal{P}} \\
 \text{rat}_X^{\mathcal{N}} \circ L \circ T_X^{\mathcal{N}} & \xrightarrow[\text{rat}_X^{\mathcal{N}} \star \varrho]{} & \text{rat}_X^{\mathcal{N}} \circ T_X^{\mathcal{N}} \circ Q \\
 & & \nearrow \alpha_X^{\mathcal{N}} \star Q \\
 & & T_X^{\text{Nori}} \circ Q
 \end{array}$$

is commutative.

The category of perverse motives $\mathcal{N}(X)$ is then defined by

$$\mathcal{N}(X) := \mathcal{N}^{\text{eff}}(X)[\mathbb{L}^{-1}].$$

Since the functor L is faithful, the canonical exact functor

$$\begin{aligned}
 \mathcal{N}^{\text{eff}}(X) &\rightarrow \mathcal{N}(X) \\
 A &\mapsto (A, 0).
 \end{aligned}$$

is also faithful. It is easy to see that the functors on the category of perverse effective motives $\text{rat}_X^{\mathcal{N}}$ and $R_X^{\mathcal{H}}$ extend to compatible faithful exact functors

$$\text{rat}_X^{\mathcal{N}} : \mathcal{N}(X) \rightarrow \mathcal{P}(X) \quad R_X^{\mathcal{H}} : \mathcal{N}(X) \rightarrow \text{MHM}(X, \mathbb{Q}).$$

Remark 7.7. — Note that by Proposition 6.6 the Abelian category $\mathcal{N}(X)$ does not depend up to an exact equivalence of categories on the choice of the functor L (and all other functors satisfying the above conditions are isomorphic).

References

- [1] Donu Arapura, *The Leray spectral sequence is motivic*, Invent. Math. **160** (2005), no. 3, 567–589. MR 2178703 (2006m:14025)
- [2] ———, *An abelian category of motivic sheaves*, Adv. Math. **233** (2013), 135–195. MR 2995668
- [3] Joseph Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I*, Astérisque (2007), no. 314, x+466 pp. (2008).
- [4] ———, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II*, Astérisque (2007), no. 315, vi+364 pp. (2008).
- [5] ———, *La réalisation étale et les opérations de Grothendieck*, Ann. Sci. École Norm. Sup. **47** (2014), no. 1, 1–141.
- [6] A. A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR 751966 (86g:32015)
- [7] Alain Bruguières, *On a tannakian theorem due to Nori*, available at <http://www.math.univ-montp2.fr/~bruguieres/docs/ntan.pdf>, 2004.

- [8] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin, 1982. MR 654325 (84m:14046)
- [9] Pierre Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448. MR 0232821 (38 #1144)
- [10] Annette Huber and Stefan Müller-Stach, *On the relation between Nori motives and Kontsevich periods*, arXiv:math/1105.0865v4, 17 April 2012.
- [11] Florian Ivorra, *Perverse, Hodge and motivic realizations of étale motives*, 2014.
- [12] J. F. Jardine, *Motivic symmetric spectra*, Doc. Math. **5** (2000), 445–553 (electronic). MR 1787949 (2002b:55014)
- [13] Marc Levine, *Mixed motives*, Handbook of K -theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 429–521. MR 2181828 (2007b:14040)
- [14] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI, 2006.
- [15] Fabien Morel and Vladimir Voevodsky, \mathbf{A}^1 -homotopy theory of schemes, Inst. Hautes Études Sci. Publ. Math. (1999), no. 90, 45–143 (2001). MR 1813224 (2002f:14029)
- [16] Madhav Nori, *Lectures at TIFR*, 32 pages.
- [17] Mirohiko Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995 (1989). MR 1000123 (90k:32038)
- [18] ———, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333. MR 1047415 (91m:14014)
- [19] ———, *Hodge conjecture and mixed motives. I*, Complex geometry and Lie theory (Sundance, UT, 1989), Proc. Sympos. Pure Math., vol. 53, Amer. Math. Soc., Providence, RI, 1991, pp. 283–303. MR 1141206 (92m:14012)
- [20] Mitsuhiro Takeuchi, *Morita theorems for categories of comodules*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), no. 3, 629–644. MR 0472967 (57 #12646)
- [21] Vladimir Voevodsky, \mathbf{A}^1 -homotopy theory, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998), no. Extra Vol. I, 1998, pp. 579–604 (electronic). MR 1648048 (99j:14018)
- [22] ———, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 188–238. MR 1764202
- [23] Jonas von Wangenheim, *Nori-Motive und Tannaka-Theorie*, Diplomarbeit, Universität Freiburg arXiv:1111.5146, 2011.

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